Basecamp for convergence of ergodic averages along subsequences

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Wrocław, Poland
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Plan

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1. *The ergodic theorem*
2. *Periodic systems*
3. *Rotation of the torus*
4. *The mean ergodic theorem*
5. *Norm-good times*
6. *Pointwise ergodic theorem*
7. *A pointwise bad time*
8. *Appendices*

8.1 What we have left out from a well equipped base camp

8.2 Maximal inequality on $\mathbb{Z}_+$

8.3 The transference

8.4 Maximal inequality and a.e. convergence

8.5 Rohlin's interval

8.6 A pointwise bad time

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The ergodic theorem

Take a cup of coffee, grab a coffee molecule x_* in it, and take repeated measurements at 1s, 2s, 3s,…, of, say, the speed of the molecule:

$$v(x_*, 1), \quad v(x_*, 2), \quad v(x_*, 3), \quad \ldots \tag{1}$$

The ergodic theorem says that the averages $\mathbb{A}_{n \in [N]} v(x_*, n) = \frac{1}{N} \sum_{n \in [N]} v(x_*, n)$ ($[N] = \{ 1, 2, \ldots, N \}$), calculated over time, converge to the average speed of all molecules in the cup:

$$\lim_N \mathop{\mathbb{A}}_{n \in [N]} v(x_*, n) = \mathop{\mathbb{A}}_{x \in ☕} v(x, 1) \tag{2}$$

So the **time averages** converge to the **space average**.

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The ergodic theorem

What's wrong with it?

$$\lim_{N} \mathop{\mathbb{A}}_{n\in[N]} v(x_*, n) = \mathop{\mathbb{A}}_{x\in ☕} v(x, 1) \qquad (3)$$

Well nothing is wrong with it, but there are two big issues:

***Question 1** (Issues with the ergodic theorem).*

(b) We do have to take the measurements exactly at equal spaced times, so like at 1s, 2s, ... What if we cannot do that (which is the likely scenario)?

(a) In general: forever. We do not address this depressing fact in these lectures.

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Question 1 *(Issues with the ergodic theorem).*

(a) How long do we need to take measurements till the time averages $\mathbf{A}_{n \in [N]} v(x_*, n)$ get close to the space one $\mathbf{A}_{x \in \text{☕}} v(x, 1)$?

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Here is what really happens when we take measurements: blue is the planned times, red is the real times.

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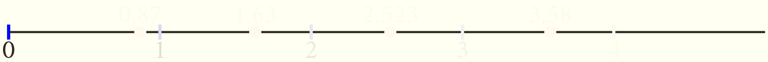

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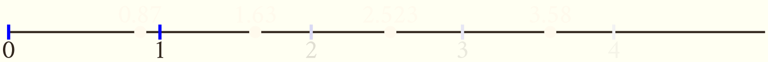

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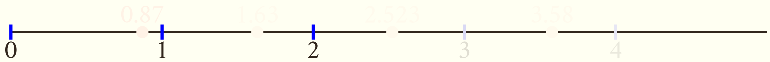

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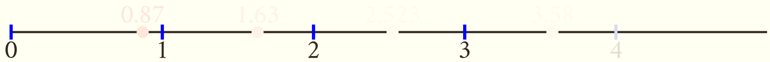

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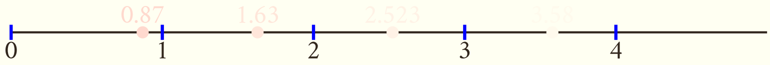

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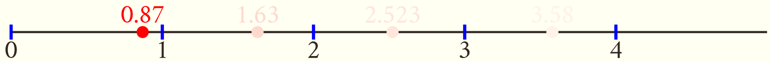

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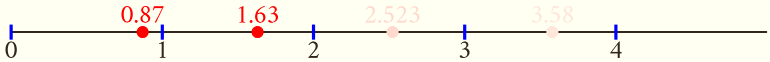

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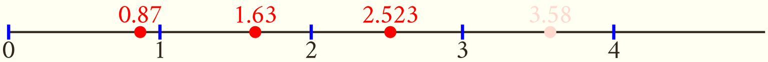

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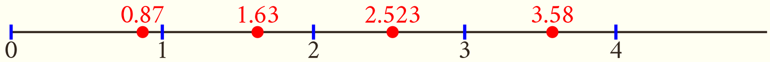

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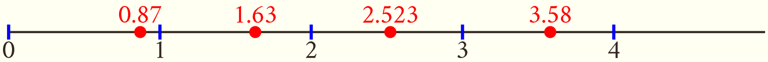

The ergodic theorem

Rationalization

Original random times placed in a rational grid

Approximation via the rational grid

Dilate the rational times to get integer times

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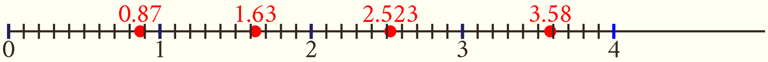

Approximation via the rational grid

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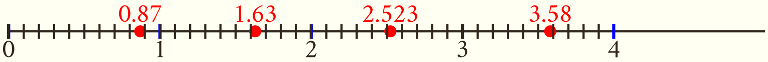

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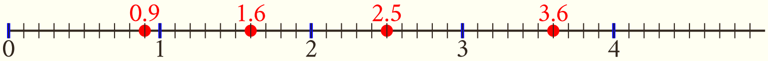

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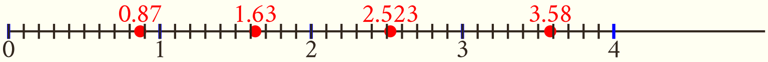

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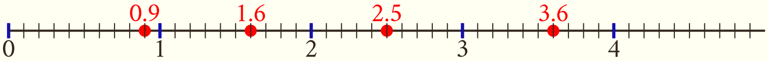

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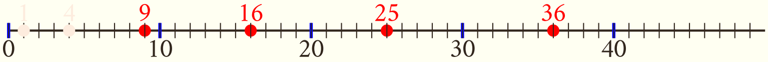

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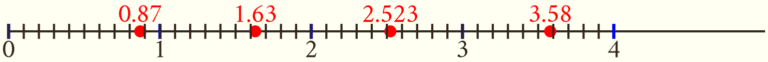

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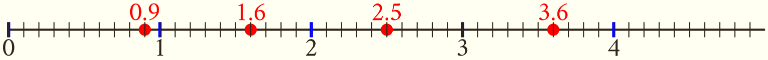

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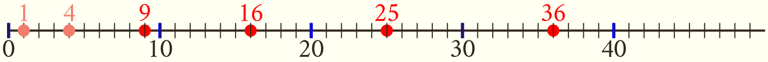

The ergodic theorem

The statement

Let $(X, \mathbf{m})$ be a probability space, so $\mathbf{m}\, X = 1$, T be a measurable, measure preserving transformation (mpt) of X, meaning that for every measurable set $A \subset X$, we have $\mathbf{m}\, T^{-1}A = \mathbf{m}\, A$.

Theorem 1 *(The ergodic theorem).*
Let $f \in L^1$.

Then we have

$$\lim_{N} \mathop{\mathbf{A}}_{n\in[N]} f(T^n x_*) \quad \text{exists for a.e. } x_* \in X \text{ and in } L^1\text{-norm} \tag{4}$$

- We didn't claim that the limit in eq. (4) is equal the "space" average which in case of a probability space would be the integral of f on the space X. This is because the limit may not be the integral of f!
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Periodic systems

A periodic system with period q is $X = [q] = \{1, 2, \ldots, q\}$ with the uniform measure $\mathbf{m} = \mathbf{m}_q$, so $\mathbf{m}_q\{j\} = 1/q$ for every $j \in [q]$. The transformation T on $[q]$ is the "shift by one $\mod q$" transformation:

$$Tj := j + 1 \pmod{q} \tag{5}$$

For this reason, we often refer to this periodic system $([q], \mathbf{m}_q, T)$ as the $\mod q$ **system**. Sometimes it's more convenient to use $[q]_0 := \{0, 1, \ldots, q-1\}$.

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A periodic system with period q is $X = [q] = \{ 1, 2, \ldots, q \}$ with the uniform measure $\mathbf{m} = \mathbf{m}_q$, so $\mathbf{m}_q\{ j \} = 1/q$ for every $j \in [q]$. The transformation T on $[q]$ is the "shift by one $\mod q$" transformation:

$$Tj \coloneqq j + 1 \pmod q \tag{5}$$

For this reason, we often refer to this periodic system $([q], \mathbf{m}_q, T)$ as the mod q **system**.
Sometimes it's more convenient to use $[q]_0 \coloneqq \{ 0, 1, \ldots, q - 1 \}$.

Periodic systems

***Problem 1** (Ergodic theorem for* $\mod q$ *systems).*

Show the ergodic theorem for $\mod q$ systems: For every $f : [q] \to \mathbb{C}$, we have

$$\lim_{N} \mathop{\mathbb{A}}_{n\in[N]} f(j_* + n) = \mathop{\mathbb{A}}_{j\in[q]} f(j) \quad \text{for every} \quad j_* \in [q] \tag{6}$$

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(b) We say $\mathbf{t}$ is **an ergodic time** or simply **ergodic** for $(X, \mathbf{m}, T)$ if it's a good time for (the ergodic) $(X, \mathbf{m}, T)$ and $\lim_N \mathbf{A}_{n\in[N]} f(T^{t_n} x_*) = \int_X f \, d\mathbf{m}$ for every $f \in L^1$.

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Periodic systems

The squares

Problem 2.

(a) [illegible]

(b) For which $q > 1$ is $\mathbb{S}$ ergodic for [illegible] mod q?

Hint:

Periodic systems

The squares

Problem 2.

(a) Show that the squares $\mathbb{S} = (n^2)_{n \in \mathbb{N}}$ is good for mod q for every q. (Show rate of convergence)

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Periodic systems

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Periodic systems

Bad times

Here is a bad sequence $\mathbf{t} = (t_n)$ for the $\bmod 2$ system: from an interval of the form $[(2k)!, (2k+1)!)$ put all the **even** numbers into $\mathbf{t}$ and from an interval of the form $[(2k-1)!, (2k)!)$ put all the **odd** numbers into $\mathbf{t}$.

In formulas, let O denote the set of odd numbers and E the set of even ones. Then define $\mathbf{t}$ (as a set) by

$$\mathbf{t} \coloneqq \bigcup_{k \in \mathbb{N}} \Big(\mathrm{O} \cap [(2k-1)!, (2k)!) \Big) \cup \Big(\mathrm{E} \cap [(2k)!, (2k+1)!) \Big) \tag{7}$$

Problem 4 *(Bad sequence for $\bmod 2$).*
Show that the sequence $\mathbf{t}$ defined in eq. (7) has the properties:

$$\limsup_{N} \mathop{\mathbb{A}}_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 1 \quad \text{for every} \quad j_* \in [2] \tag{8}$$

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Periodic systems

Bad times 2

Problem 5 *(Bad sequence for mod q).*
Let $q \geq 2$.
Construct a sequence $\mathbf{t} = (t_n)$ so that

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Hint:
For $r \in [q]$ denote $\mathrm{R}_r \coloneqq \{\, n \in \mathbb{N} \;:\; n \equiv r \pmod{q} \,\}$ and define

$$\mathbf{t} \coloneqq \bigcup_{k \in \mathbb{N}} \bigcup_{r \in [q]} \Big(\mathrm{R}_r \cap [(kq + r)!, (kq + r + 1)!) \Big) \tag{12}$$

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For $r \in [q]$ denote $R_r \coloneqq \{ n \in \mathbb{N} : n \equiv r \pmod q \}$ and define

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Periodic systems

Bad times 3

Problem 6 *(Bad sequence for all mod q).*
Construct a sequence $\mathbf{t} = (t_n)$ so that in **every** mod q system we have

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Periodic systems

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Problem 6 *(Bad sequence for all* $\bmod q$*).*
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Rotation of the torus

- The torus $\mathbb{T}$ is defined as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We identify $\mathbb{T}$ with the closed unit interval $[0, 1]$, with 0 and 1 glued together.
- We denote by λ the Lebesgue-Haar probability measure on $\mathbb{T}$.
- We use Weyl's notation $e(\theta) := e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$.
- For a real number x we define [illegible]

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- For a real number α we define the transformation $T = T_\alpha$ of $\mathbb{T}$ by

$$T\theta \coloneqq \{\theta + \alpha\} \quad \text{for every} \quad \theta \in \mathbb{R} \tag{15}$$

where $\{x\}$ is the **fractional part** of $x \in \mathbb{R}$, so if $n \leq x < n + 1$ for some $n \in \mathbb{Z}$, then $\{x\} = x - n$. Following *Weyl*, we may write $x \mod 1$ instead of $\{x\}$.
- We call T_α the **shift by** α or **rotation by** α, the latter term being justified by viewing the effect of T_α under the map $\theta \mapsto \mathbf{e}(\theta)$: $\mathbf{e}(T_\alpha\theta) = \mathbf{e}(\theta)\,\mathbf{e}(\alpha)$, and multiplication by $\mathbf{e}(\alpha)$ rotates the complex plane (hence the unit circle) by the angle $2\pi\alpha$.

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Rotation of the torus

Rational α

The case of rotating by a rational $\alpha = \frac{a}{q}$ can be seen as the shift by a in the $\mod q$ system, thus for rational α we only have this to say:

Problem 7 *(Rational rotation is not ergodic).*
Show that the transformation $T_{a/q}$ is not ergodic with respect to λ.

Rotation of the torus

Rational α

The case of rotating by a rational $\alpha = \frac{a}{q}$ can be seen as the shift by a in the $\mod q$ system, thus for rational α we only have this to say:

***Problem** 7 (Rational rotation is not ergodic).*
Show that the transformation $T_{a/q}$ is not ergodic with respect to λ.

Rotation of the torus

Irrational α

Lemma 1 *(Weyl's lemma).*
Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

Then we have $\lim_N \mathbf{A}_{n\in[N]_0} \mathbf{e}(n\alpha) = 0$.

The proof is an application of the geometric sequence sum-formula, according to which $\mathbf{A}_{n\in[N]_0} \mathbf{e}(n\alpha) = \frac{1}{N}\frac{\mathbf{e}(N\alpha)-1}{\mathbf{e}(\alpha)-1}$. Since $\mathbf{e}(\alpha) - 1 \neq 0$, letting $N \to \infty$ proves the lemma.

Problem 8 *(Irrational rotation is ergodic).*
Show that the rotation of $\mathbb{T}$ by an irrational number α is ergodic with respect to λ.

Hint:
Use first lemma 1 with $\alpha = p\alpha$ for every $p \in \mathbb{Z}\setminus\{0\}$, then use the density of the trigonometric polynomials in $L^1(\mathbb{T}, \lambda)$.

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The mean ergodic theorem

Let $(X, \mathbf{m}, T)$. The main observation (by *Koopman*) is that the linear transformation $\mathcal{T}$ of L^2 defined by

$$\mathcal{T}f(x) \coloneqq f(Tx) \quad \text{for every} \quad f \in L^2 \tag{16}$$

is a linear **isometry**, meaning that $\|\mathcal{T}f\|_{L^2} = \|f\|_{L^2}$. Since L^2 is a Hilbert space, the following is a general version of what we need.

Theorem 2 *(Mean ergodic theorem for Hilbert space).*
Let $\mathcal{T}$ be a **contraction** of a Hilbert space H, so $\|\mathcal{T}f\|_H \le \|f\|_H$.

Then

$$\mathcal{P}f \coloneqq \lim_N \mathop{\mathbf{A}}_{n\in[N]_0} \mathcal{T}^n f \tag{17}$$

exists in the norm of H for every $f \in H$. The transformation $\mathcal{P}$ is $\mathcal{T}$ invariant: $\mathcal{T}\mathcal{P} = \mathcal{P}$.

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Hilbert space proof

The idea of Riesz for proving the mean ergodic theorem is that two classes of functions in H, for which the the theorem is close to obvious, happen to span all of H.

The first class of functions is the $\mathcal{T}$ invariant ones: $\mathcal{T}f = f$. Denoting the set of **$\mathcal{T}$-invariant** functions by I, we have

$$\lim_{N} \mathop{\mathbf{A}}_{n\in[N]_0} \mathcal{T}^n f = f \quad \text{for} \quad f \in I \tag{18}$$

The other set of functions is the set C of **coboundaries**; these are functions f which can be written in the form $f = \mathcal{T}g - g$ for some $g \in H$. For such an f we have a collapsing sum: $\sum_{n\in[N]_0} \mathcal{T}^n f = \mathcal{T}^N g - g$, thus

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The punchline is the observation that H is the orthogonal sum of I and $\overline{C}$, $H = I \perp \overline{C}$. What needs to be proven is that if f is orthogonal to every coboundary then it must be $\mathcal{T}$-invariant. A picture will make this obviously true: (next slide, please)

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Hilbert space proof 2

Again, what we are proving is that if f is orthogonal to every coboundary, then it must be $\mathcal{T}$-invariant, so $\mathcal{T}f = f$.

If $f \perp \mathrm{C}$ then, in particular, $f \perp \mathcal{T}f - f$. Here is the picture.

In this right angle triangle, if the length of $\mathcal{T}f - f$ was positive then, by the Pythagorean theorem, $\mathcal{T}f$ would be **strictly longer** than the length of f, but $\mathcal{T}$ is an contraction, so $\|\mathcal{T}f\| \leq \|f\|$. So we must have $\|\mathcal{T}f - f\| = 0$, implying $\mathcal{T}f = f$.

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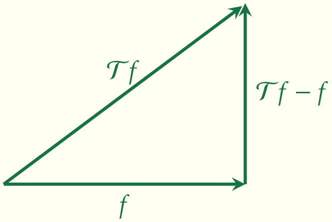

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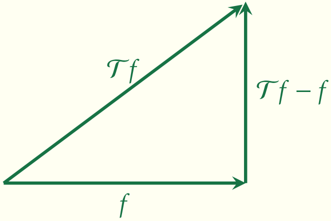

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Norm-good times

Lemma 2 *(Herglotz's spectral theorem (baby spectral theorem)).*
Let $\mathcal{U}$ be a unitary transformation of the Hilbert space H, and let $f \in \mathrm{H}$.

Then there is a Borel measure $\mu = \mu_f$ on $\mathbb{T}$ so that

$$(\mathcal{U}^n f, f) = \int_{\mathbb{T}} \mathbf{e}^{-n} \,\mathrm{d}\mu \quad \text{for every} \quad n \in \mathbb{Z} \tag{20}$$

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Recall: $(\mathcal{U}^n f, f) = (f, \mathcal{U}^{-n} f) = \int_{\mathbb{T}} \mathbf{e}^{-n} \, \mathrm{d}\mu_f$.

Problem 9.
Show (or just observe) the following for $f \in \mathrm{H}$:

(a) $\|f\|_{\mathrm{H}}^2 = \mu_f \mathbb{T}$.

(b) $\left\| \mathbf{A}_{n \in [N]} \mathcal{U}^n f \right\|_{\mathrm{H}}^2 = \int_{\mathbb{T}} \left| \mathbf{A}_{n \in [N]} \mathbf{e}^{-n} \right|^2 \mathrm{d}\mu_f$.

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(b) Calculation using baby spectral theorem

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Recall: $(\mathcal{U}^n f, f) = (f, \mathcal{U}^{-n} f) = \int_{\mathbb{T}} \mathbf{e}^{-n} \, \mathrm{d}\mu_f$.

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Show (or just observe) the following for $f \in \mathrm{H}$:

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Definition 2 *(Norm-good times).*
Let $(X, \mathbf{m}, T)$ be a mps. We say the time $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$ is **norm-good for** $(X, \mathbf{m}, T)$ if $\lim_N \mathbf{A}_{n\in[N]} f \circ T^{t_n}$ exists in $\|\|_{L^2(X)}$ for every $f \in L^2(X)$.
We say the time $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$ is **norm-good** if it's norm-good for every mps.

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The time $\mathbf{t} = (t_n)_{n\in\mathbb{N}}$ is norm-good iff it's norm good for every rotation of the torus $\mathbb{T}$:

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***Theorem 4** (Pointwise ergodic theorem).*
Let $(X, \mathbf{m}, T)$ be a measure preserving system and $f \in L^1(X)$.

Then $\lim_N \mathbf{A}_{n \in [N]_0} f(T^n x)$ exists for $\mathbf{m}$-a.e. $x \in X$.

***Corollary 2** (Irrational rotations).*
Let α be irrational.

Then $\lim_N \mathbf{A}_{n \in [N]_0} f(\theta + n\alpha) = \int_{\mathbb{T}} f \, d\lambda$ for $f \in L^1(\mathbb{T}, \lambda)$ and λ-a.e. θ.

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The steps of proving the pointwise ergodic theorem are

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The dense class is the same as in case of the mean ergodic theorem: those $f \in L^2(X)$ which can be written as a sum of a T-invariant function and a coboundary. The only question is a coboundary, so when $f = g \circ T - g$ for some $g \in L^2$. Since $\mathbf{A}_{n \in [N]_0} f(T^n x) = \frac{1}{N} \cdot (g(T^N x) - g(x))$,

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The dense class is the same as in case of the mean ergodic theorem: those $f \in L^2(X)$ which can be written as a sum of a T-invariant function and a coboundary. The only question is a coboundary, so when $f = g \circ T - g$ for some $g \in L^2$. Since $\mathbf{A}_{n\in[N]_0} f(T^n x) = \frac{1}{N} \cdot \left(g(T^N x) - g(x)\right)$,

$$\int_X \sum_N \left| \mathbf{A}_{n\in[N]_0} f(T^n x) \right|^2 \leq \sum_N \frac{1}{N^2} \cdot 4 \cdot \|g\|^2_{L^2(X)} < \infty \tag{22}$$

so $\lim_N \mathbf{A}_{n\in[N]_0} f(T^n x) = 0$ for $\mathbf{m}$-a.e. $x \in X$.

Pointwise ergodic theorem

Maximal inequality

The pointwise convergence from a dense set of L^1 is extended to all of L^1 by a **maximal inequality**:

$$\mathbf{m}\left\{ x \in X \,:\, \sup_N \left| \mathop{\mathbf{A}}_{n\in[N]_0} f(T^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{L^1(X)} \quad \text{for every} \quad f \in L^1(X), \eta > 0 \tag{23}$$

The function $\sup_N \left|\mathbf{A}_{n\in[N]_0} f(T^n x)\right|$ is called the **maximal function of f**. Note that for any $g \in L^1(X)$ we have Markov's inequality $\mathbf{m}\{ x \in X \,:\, \sup_N |g(x)| > \eta \} \le \frac{1}{\eta}\|g\|_{L^1(X)}$. In general, the maximal function is not integrable, though it satisfies the **weak inequality** in eq. (23).

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Scheme to prove the maximal inequality

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we can assume that $f \geq 0$ and $\eta = 1$. The proof of eq. (24) is in two steps:

- Prove the special case when the mps is $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ with the counting measure and T is the shift by 1, so $Tj = j + 1$. We thus first prove

$$\#\left\{ j \in \mathbb{Z}_+ : \max_{N\in[M]} \mathop{\mathbb{A}}_{n\in[N]_0} F(j+n) > 1 \right\} \le \sum_j F(j) \tag{25}$$

  for $F : \mathbb{Z}_+ \to \mathbb{R}_+$ of finite support and $M \in \mathbb{N}$.
- Use Calderón's transference to show the inequality in a general mps.

Basecamp for convergence

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A transformation T of a measure space $(X, \mathbf{m})$ is **aperiodic** if

$$\mathbf{m}\{ x \in X : T^q x = x \quad \text{for some} \quad q \in \mathbb{N} \} = 0 \tag{26}$$

Recall that earlier you constructed a time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ so that for every $q \in \mathbb{N}$ $\limsup_N \mathbf{A}_{n \in [N]} \mathbb{1}_{\{0\}}(r + t_n) = 1$ for every $r \in [q]$.

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Let $(X, \mathbf{m}, T)$ be an aperiodic mps and $\varepsilon > 0$.
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Appendix

Maximal inequality on $\mathbb{Z}_+$

We want to prove

$$\#\left\{ j \in \mathbb{Z}_+ : \max_{N\in[M]} \mathop{\mathbf{A}}_{n\in[N]_0} F(j+n) > 1 \right\} \le \sum_j F(j) \quad \text{for } F : \mathbb{Z}_+ \to \mathbb{R}_+ \text{ of finite support and } M \in \mathbb{N} \tag{27}$$

- Let $S = S_1 := \left\{ j \in \mathbb{Z}_+ : \sum_{n\in[N]_0} F(j+n) > N \quad \text{for some} \quad N \in [M] \right\}$. Let j_1 be the smallest element of S_1. Then for an $N_1 \in [M]$ we have $\sum_{n\in[N_1]_0} F(j_1+n) > N_1$ which is

$$N_1 < \sum_{j\in[j_1, j_1+N_1-1]} F(j) \tag{28}$$

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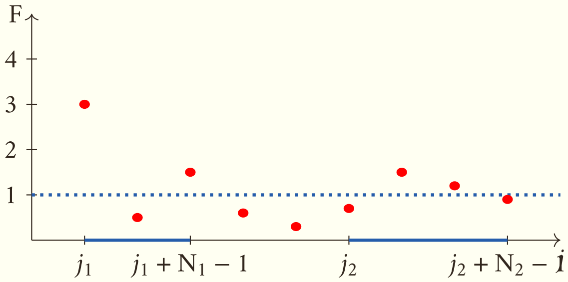

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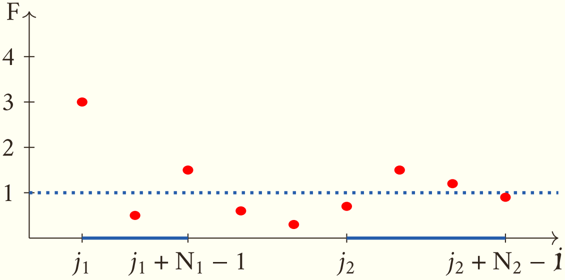

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Denote $S \coloneqq \{ x \in X : \max_{N\in[M]} \mathbf{A}_{n\in[N]_0} f(T^n x) > 1 \}$. We want to show $\mathbf{m}\, S \le \int_X f \,\mathrm{d}\mathbf{m}$. Fix $x \in X$ and let $J \in \mathbb{N}$, $J > M$ and define $F : [J]_0 \to \mathbb{R}_+$ by

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Figure: Transference from $[J]_0$ to any measure preserving system

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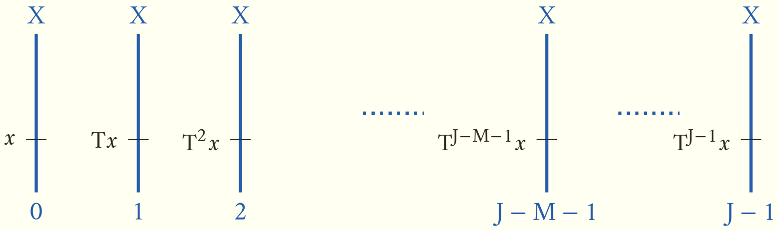

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Maximal inequality and a.e. convergence

How does the maximal inequality

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and a.e. convergence for an L^1-dense class imply a.e. convergence for all of L^1?

- Fix $f \in L^1(X)$, let $\varepsilon > 0$, and let g be from dense class for which $\lim_N \mathbf{A}_{n\in[N]_0} g(T^n x)$ exists a.e. and approximates f ε-closely: $f = g + h$ where $\|h\|_{L^1(X)} < \varepsilon$.

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- Since $\omega h(x) \leq 2 \sup_N \left| \mathbf{A}_{n \in [N]_0} h(x) \right|$, by the maximal inequality, we get $\mathbf{m}\{ x \in X : \omega f(x) > \eta \} \leq \mathbf{m}\{ x \in X : 2 \sup_N \left| \mathbf{A}_{n \in [N]_0} h(x) \right| > \eta \} \leq \frac{2}{\eta} \|h\|_{L^1(X)}$.

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- Since $\|h\|_{L^1(X)} < \varepsilon$ and $\varepsilon > 0$ was arbitrary, we get that $\mathbf{m}\{ x \in X : \omega f(x) > \eta \} = 0$. Since this is true for every $\eta > 0$, we get that $\omega f(x) = 0$ for $\mathbf{m}$-a.e. $x \in X$.

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Maximal inequality and a.e. convergence

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Theorem 5 *(Rohlin's interval with error).*
Let T be an invertible, bimeasurable, measure preserving, aperiodic transformation of the measure space $(X, \mathbf{m})$. Let $[a, b] \subset \mathbb{Z}$ be an interval and let $\varepsilon > 0$.

Then there is a measurable $B \subset X$ so that the sets in the family $\{T^j B : j \in [a, b]\}$ are pairwise disjoint and $\mathbf{m}\left(\cup_{j\in[a,b]} T^j B\right)^c < \varepsilon$.

The set $E := \left(\cup_{j\in[a,b]} T^j B\right)^c$ is called the **error set**, and we talk about *Rohlin's interval* $[a, b]$ *with error* ε. A bit more casually, we say *we represent the interval* $[a, b]$ *in a mps*. In the literature they talk about *Rohlin's* ***tower*** *of height* H *with error* ε and they mean representing the interval $[0, H)$.

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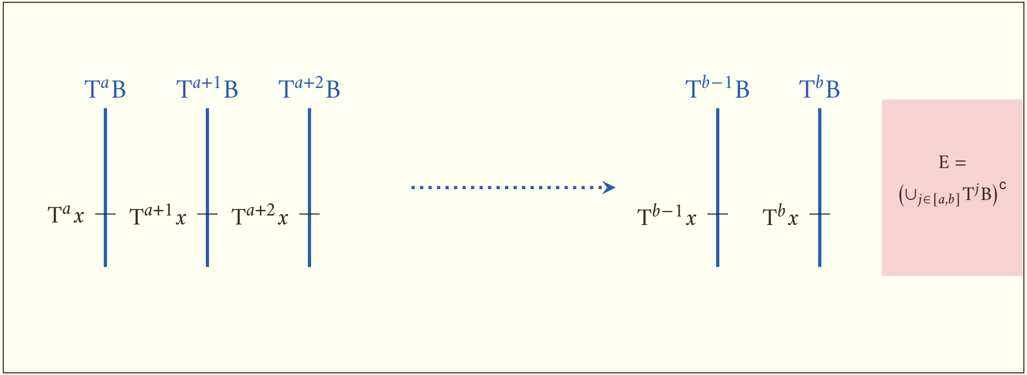
Rohlin's interval $[a, b]$ and error:



$$X = T^{[a,b]} B \cup E = \left(\cup_{j \in [a,b]} T^j B\right) \cup E$$

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Let $(X, \mathbf{m}, T)$ be an aperiodic mps and $\varepsilon > 0$.
Show that there is $A \subset X$ with $\mathbf{m}\, A < \varepsilon$ and $\limsup_N \mathbf{A}_{n\in[N]} \mathbb{1}_A(T^{t_n}x) = 1$ for a.e. $x \in X$.

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