Basecamp for convergence of ergodic averages along subsequences

Máté Wierdl

University of Memphis

Wrocław, Poland September 6, 2024 Basecamp for convergence

Máté Wierdl

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$$V(\lambda_0, 1), \quad V(\lambda_0, 2), \quad U(\lambda_0, 3), \quad \dots$$

$$\lim_{N} \underbrace{\mathbf{A}}_{n \in [N]} v(x_n, n) = \underbrace{\mathbf{A}}_{11} \underbrace{\mathbf{A}}_{x \in \mathbb{Z}} v(x, 1) \tag{2}$$

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Take a cup of coffee

$$(x_*, 1), \quad \nu(x_*, 2), \quad \nu(x_*, 3), \quad \dots$$

$$\lim_{N} \underbrace{\mathbf{A}}_{n \in [N]} v(x_n, n) = \underbrace{\mathbf{A}}_{\substack{\{i\}\}\\ x \in \mathbb{Z}^{2}}} v(x, 1)$$

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Take a cup of coffee , grab a coffee molecule x_* in it, and take repeated

$$v(x_*, 1), v(x_*, 2), v(x_*, 3), \dots$$
 (1)

$$\lim_{N} \underbrace{\mathbf{A}}_{n \in [N]} v(x_n, n) = \underbrace{\mathbf{A}}_{111} v(x, 1) \tag{2}$$

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, grab a coffee molecule x_* in it, and take repeated Take a cup of coffee measurements at 1s, 2s, 3s,..., of, say, the speed of the molecule:

$$v(x_*, 1), v(x_*, 2), v(x_*, 3), \dots$$
 (1)

$$\lim_{N} \underbrace{\mathbf{A}}_{n \in [N]} v(x_*, n) = \underbrace{\mathbf{A}}_{\{\}\}} v(x, 1)$$
 (2)

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$$v(x_*, 1), v(x_*, 2), v(x_*, 3), \dots$$
 (1)

The ergodic theorem says that the averages $\mathbf{A}_{n \in [N]} v(x_*, n) = \frac{1}{N} \sum_{n \in [N]} v(x_*, n)$ $([N] = \{1, 2, ..., N\})$, calculated over time, converge to the average speed of all molecules in the cup:

$$\lim_{N} \mathbf{A}_{n \in [N]} v(x_*, n) = \mathbf{A}_{\underset{x \in \mathbb{Z}^2}{\{\}\}}} v(x, 1) \tag{2}$$

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$$\lim_{N} \mathbf{A}_{n \in [N]} v(x_*, n) = \mathbf{A}_{\underset{x \in \mathbb{Z}^{D}}{\{\xi\}\}}} v(x, 1)$$
 (2)

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Take a cup of coffee , grab a coffee molecule x_* in it, and take repeated measurements at 1s, 2s, 3s,..., of, say, the speed of the molecule:

$$v(x_*, 1), v(x_*, 2), v(x_*, 3), \dots$$
 (1)

The ergodic theorem says that the averages $\mathbf{A}_{n \in [N]} v(x_*, n) = \frac{1}{N} \sum_{n \in [N]} v(x_*, n)$ $([N] = \{1, 2, ..., N\})$, calculated over time, converge to the average speed of all molecules in the cup:

$$\lim_{N} \mathbf{A}_{n \in [N]} v(x_*, n) = \mathbf{A}_{\underbrace{\S\S\S}} v(x, 1)$$
 (2)

So the **time averages** converge to the **space average**.

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$$\lim_{N} \mathbf{A}_{n \in [N]} v(x_*, n) = \mathbf{A}_{x \in \Sigma} v(x, 1)$$

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(3)

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What's wrong with it?

$$\lim_{N} \mathbf{A}_{n \in [N]} v(x_*, n) = \mathbf{A}_{\underset{x \in \mathbb{C}^{N}}{\{\xi\}\}}} v(x, 1)$$
(3)

Well nothing is wrong with it, but there are two big issues:

Question 1 (Issues with the ergodic theorem).

(a) How long do we need to take measurements till the time averages $\mathbf{A}_{n \in [N]} v(x_*, n)$ get

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What's wrong with it?

$$\lim_{N} \mathbf{A}_{n \in [N]} v(x_*, n) = \mathbf{A}_{\underset{x \in \mathcal{D}}{\{\xi\}\}}} v(x, 1)$$
(3)

Well nothing is wrong with it, but there are two big issues:

Question 1 (Issues with the ergodic theorem).

- (a) How long do we need to take measurements till the time averages $\mathbf{A}_{n \in [N]} v(x_*, n)$ get close to the space one $\mathbf{A}_{n \in [N]} v(x, 1)$?
- (b) We do have to take the measurements **exactly** at equal spaced times, so like at 1s, 2s,.... What if we cannot do that (which is the likely scenario)?
- (a) In general: Forever. We do not address this depressing fact in these lectures.
- (b) This is also depressing, but this is our main topic, so we do address it, and, as usual in math, we'll try to squeeze something good out of this impossible situation.

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What's wrong with it?

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What's wrong with it?

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(3)

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- (a) How long do we need to take measurements till the time averages $\mathbf{A}_{n \in [N]} v(x_*, n)$ get close to the space one $\mathbf{A}_{x_* \in [N]} v(x, 1)$?
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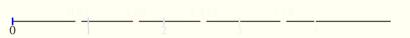
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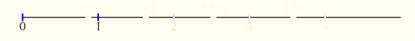
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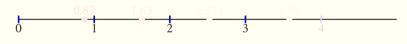
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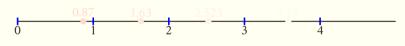
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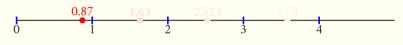
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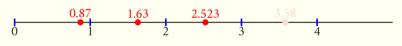
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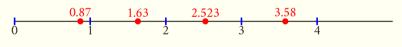
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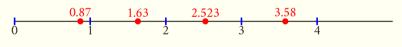
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Approximation via the rational grid



Dilate the rational times to get integer times



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The statement

Let (X, \mathbf{m}) be a probability space, so $\mathbf{m} X = 1$, T be a measurable, measure preserving transformation (mpt) of X, meaning that for every measurable set $A \subset X$, we have $\mathbf{m} T^{-1}A = \mathbf{m} A$.

Theorem 1 (The ergodic theorem). Let $f \in L^1$.

Then we hav

- We didn't claim that the limit in eq. (4) is equal the "space" average which in case of a probability space would be the integral of f on the space X. This is because the limit may not be the integral of f!
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$$\lim_{N} \mathbf{A}_{n \in [N]} f(T^{n} x_{*}) \quad \text{exists for a.e. } x_{*} \in X \text{ and in } L^{1} \text{-norm}$$

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- We say that the mpt T is **ergodic** if the limit in the ergodic theorem is equal to the integral.
- In case T is ergodic and the space is a finite, but not necessarily probability space, the integral needs to be replaced by the integral average $\frac{1}{\sqrt{N}} \int_{N} f \, d\mathbf{m}$.

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- In case T is ergodic and the space is a finite, but not necessarily probability space, the integral needs to be replaced by the integral average $\frac{1}{2\pi} \int_{\mathbb{R}^2} f d\mathbf{m}$.

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A periodic system with period q is $X = [q] = \{1, 2, \dots, q\}$ with the uniform measure $\mathbf{m} = \mathbf{m}_q$, so $\mathbf{m}_{q}\{j\} = 1/q$ for every $j \in [q]$. The transformation T on [q] is the "shift by one mod a"

$$\Gamma j \coloneqq j + 1 \pmod{q} \tag{5}$$

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$$Tj := j + 1 \pmod{q} \tag{5}$$

For this reason, we often refer to this periodic system ([q], \mathbf{m}_q , T) as the mod q system. Sometimes it's more convenient to use $[q]_0 \coloneqq \{0, 1, ..., q-1\}$.

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$$Tj \coloneqq j + 1 \pmod{q}$$

For this reason, we often refer to this periodic system ([q], \mathbf{m}_q , T) as the $\mathbf{mod}\ q$ system. Sometimes it's more convenient to use $[q]_0 := \{0, 1, \dots, q-1\}$.

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$$\lim_{N} \mathbf{A}_{n \in [N]} f(j_* + n) = \mathbf{A}_{j \in [q]} f(j) \quad \text{for every} \quad j_* \in [q]$$
 (6)

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The ergodic theorem

Problem 1 (Ergodic theorem for mod q systems).

Show the ergodic theorem for mod q systems: For every $f:[q] \to \mathbb{C}$, we have

$$\lim_{N} \mathbf{A}_{n \in [N]} f(j_* + n) = \mathbf{A}_{j \in [q]} f(j) \quad \text{for every} \quad j_* \in [q]$$
 (6)

Hint:

Assume that f is the indicator of a point in [q].

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Good and ergodic times

Definition 1 (Good and ergodic times).

Let $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ be a sequence of integers, and (X, \mathbf{m}, T) a mps.

(a) We say t is a good time or simply good for (X, \mathbf{m}, T) if $\lim_{N \to \infty} \mathbf{A}_{n \in [N]} f(T^{t_n} x_*)$ exists for every $f \in L^1$.

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- (b) We say \mathbf{t} is an ergodic time or simply ergodic for (X, \mathbf{m}, T) if it's a good time for (the ergodic) (X, **m**, T) and $\lim_{N} \mathbf{A}_{n \in [N]} f(T^{t_n} x_*) = \int_{Y} f d\mathbf{m}$ for every $f \in L^1$.

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So \mathbb{N} is ergodic for $\mod q$ systems.

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The squares

Problem 2.

(a) Show that the squares $\mathbb{S} = (n^2)_{n \in \mathbb{N}}$ is good for mod q for every q. (Show rate of convergence)

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- (b) For which q > 1 is \mathbb{S} ergodic for mod q?

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- (b) For which q > 1 is \mathbb{S} ergodic for $\mod q$?

Hint

(a) Write *n* in the form n = tq + r for some integers *t* and $0 \le r < q$

(b) Only q = 2.

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The primes

Problem 3.

(a) Show that the sequence \mathbb{P} of primes is good for mod q for every q. (Show rate of convergence)

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The primes

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- (b) For which q > 1 is \mathbb{P} ergodic for $\mod q$?

Hint

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- (a) You need to look into the literature for the prime number theorem in arithmetic progressions, including rate of convergence.
- (b) None.

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$$\mathbf{t} := \bigcup_{k \in \mathbb{N}} \{ O \cap [(2k - 1)] \}$$

$$\lim_{N} \inf \mathbf{A} 1_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [2]$$

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Bad times

Here is a bad sequence $\mathbf{t} = (t_n)$ for the mod 2 system: from an interval of the form

$$= \bigcup_{k \in \mathbb{N}} \left(O \cap \left[(2k-1)!, (2k)! \right) \right) \cup \left(\mathbb{E} \cap \left[(2k)!, (2k+1)! \right) \right)$$

$$\lim_{n \to \infty} A = \lim_{n \to \infty} A =$$

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Bad times

Here is a bad sequence $\mathbf{t} = (t_n)$ for the mod 2 system: from an interval of the form [(2k)!, (2k+1)!) put all the **even** numbers into \mathbf{t} and from an interval of the form [(2k-1)!, (2k)!) put all the **odd** numbers into \mathbf{t} .

by

$$\mathbf{t} := \bigcup_{k \in \mathbb{N}} \left(O \cap \left[(2k-1)!, (2k)! \right) \right) \cup \left(E \cap \left[(2k)!, (2k+1)! \right) \right) \tag{7}$$

Problem 4 (Bad sequence for mod 2).

Show that the sequence **t** defined in eq. (7) has the properties:

$$\min_{N} \bigwedge_{i=1}^{n} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [2]$$
 (9)

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Bad times

Here is a bad sequence $\mathbf{t} = (t_n)$ for the mod 2 system: from an interval of the form [(2k)!, (2k+1)!] put all the **even** numbers into **t** and from an interval of the form [(2k-1)!,(2k)!] put all the **odd** numbers into **t**.

In formulas, let O denote the set of odd numbers and E the set of even ones. Then define t (as a set)

$$\mathbf{t} \coloneqq \bigcup_{k \in \mathbb{N}} \left(O \cap \left[(2k-1)!, (2k)! \right) \right) \cup \left(\mathbb{E} \cap \left[(2k)!, (2k+1)! \right) \right) \tag{7}$$

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In formulas, let O denote the set of odd numbers and E the set of even ones. Then define t (as a set) by

$$\mathbf{t} := \bigcup_{k \in \mathbb{N}} \left(\mathcal{O} \cap \left[(2k-1)!, (2k)! \right) \right) \cup \left(\mathcal{E} \cap \left[(2k)!, (2k+1)! \right) \right)$$
 (7)

$$\liminf_{N} \bigwedge_{t \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [2]$$
 (9)

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Here is a bad sequence $\mathbf{t} = (t_n)$ for the mod 2 system: from an interval of the form

$$\mathbf{t} \coloneqq \bigcup_{k \in \mathbb{N}} \left(\mathcal{O} \cap \left[(2k-1)!, (2k)! \right) \right) \cup \left(\mathcal{E} \cap \left[(2k)!, (2k+1)! \right) \right)$$

Problem 4 (Bad sequence for mod 2).

Show that the sequence **t** defined in eq. (7) has the properties:

$$\limsup_{N} \mathbf{A}_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 1 \quad \text{for every} \quad j_* \in [2]$$

$$\liminf_{N} \mathbf{A}_{\{0\}} [j_* + t_n] =$$

$$\lim_{N} \inf_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [2]$$
 (9)

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$$\limsup_{N} A_{n \in [N]} 1_{\{}$$

 $\limsup A \mathbb{1}_{\{0\}}(j_* + t_n) = 1 \text{ for every } j_* \in [q]$

 $\liminf_{N} A_{\{0\}} (j_* + t_n) = 0 \text{ for every } j_* \in [q]$

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Bad times 2

Problem 5 (Bad sequence for mod q).

Let $a \geq 2$.

Construct a sequence $\mathbf{t} = (t_n)$ so that

$$\limsup \mathbf{A} \mathbb{1}_{\{0\}}$$

$$\limsup \mathbf{A} \mathbb{1}_{\{0\}}$$

$$\limsup \mathbf{A} \mathbb{1}_{\{0\}}$$

$$\limsup_{N} \mathbf{A}_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 1 \quad \text{for every} \quad j_* \in [q]$$

$$\liminf_{N} \bigwedge_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [q]$$

$$\prod_{N} \prod_{n \in [N]} \mathbb{I}\{0\}() * + \mathfrak{l}_n) = 0$$

 $\mathbf{t} \coloneqq \bigcup \left(\mathbb{R}_r \cap [(kq+r)!, (kq+r+1)!) \right)$

for every
$$j_* \in [q]$$

$$j_* \in [q]$$

$$[q] (11)$$

(10)

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Let $a \geq 2$.

Hint:

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Problem 5 (Bad sequence for mod q).

Construct a sequence $\mathbf{t} = (t_n)$ so that

For $r \in [q]$ denote $R_r := \{ n \in \mathbb{N} : n \equiv r \pmod{q} \}$ and define

$$1(\alpha)(i_1+t_2)=1$$

$$\limsup_{N} \bigwedge_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 1 \quad \text{for every} \quad j_* \in [q]$$

 $\mathbf{t} := \bigcup_{k \in \mathbb{N}} \bigcup_{r \in [q]} \left(\mathbf{R}_r \cap [(kq+r)!, (kq+r+1)!) \right)$

$$\liminf_{N} \bigwedge_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [q]$$

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$$\liminf_{N} \bigwedge_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [q]$$

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Bad times 3

Problem 6 (Bad sequence for all mod q).

Construct a sequence $\mathbf{t} = (t_n)$ so that in **every** mod q system we have

$$\limsup_{N} \bigwedge_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 1 \quad \text{for every} \quad j_* \in [q]$$

$$\liminf_{n \to \infty} \mathbf{A} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every}$$

$$\liminf_{N} \bigwedge_{n \in [N]} \mathbb{1}_{\{0\}}(j_* + t_n) = 0 \quad \text{for every} \quad j_* \in [q]$$

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- ▶ The torus T is defined as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We identify T with the closed unit interval [0, 1], with 0 and 1 glued together.
- We denote by λ the Lebesgue-Haar probability measure on ${\mathbb T}$
- ▶ We use Weyl's notation $e(\theta) := e^{2\pi i \theta}$ for $\theta \in \mathbb{R}$.
- For a real number a we define the transformation $T=T_a$ of

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- ▶ The torus \mathbb{T} is defined as $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We identify \mathbb{T} with the closed unit interval [0,1], with 0 and 1 glued together.
- \blacktriangleright We denote by λ the Lebesgue-Haar probability measure on \mathbb{T} .

$$\Gamma 0 := \{ 0 + \alpha \} \quad \text{for every} \quad 0 \in \mathbb{R}$$
 (15)

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$$\Gamma\theta \coloneqq \left\{\theta + \alpha\right\} \quad \text{for every} \quad \theta \in \mathbb{R}$$
 (15)

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$$T\theta := \{\theta + \alpha\} \quad \text{for every} \quad \theta \in \mathbb{R}$$
 (15)

where $\{x\}$ is the **fractional part** of $x \in \mathbb{R}$, so if $n \le x < n+1$ for some $n \in \mathbb{Z}$, then $\{x\} = x - n$. Following *Weyl*, we may write $x \mod 1$ instead of $\{x\}$.

▶ We call T_{α} the **shift by** α or **rotation by** α, the latter term being justified by viewing the effect of T_{α} under the map $\theta \mapsto \mathbf{e}(\theta)$: $\mathbf{e}(T_{\alpha}\theta) = \mathbf{e}(\theta) \mathbf{e}(\alpha)$, and multiplication by $\mathbf{e}(\alpha)$ rotates the complex plane (hence the unit circle) by the angle $2\pi\alpha$.

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Rational a

The case of rotating by a rational $\alpha = \frac{a}{a}$ can be seen as the shift by a in the mod q system, thus for rational α we only have this to say:

Problem 7 (Rational rotation is not ergodic).

Show that the transformation $T_{a/a}$ is not ergodic with respect to λ .

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Irrational a

Lemma 1 (Weyl's lemma). Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

Then we have $\lim_{N} \mathbf{A}_{n \in [N]_0} \mathbf{e}(n\alpha) = 0$.

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Lemma 1 (Weyl's lemma). Let $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

Then we have $\lim_{N} \mathbf{A}_{n \in [N]_0} \mathbf{e}(n\alpha) = 0$.

The proof is an application of the geometric sequence sum-formula, according to which $\mathbf{A}_{n \in [N]_0} \mathbf{e}(n\alpha) = \frac{1}{N} \frac{\mathbf{e}(N\alpha) - 1}{\mathbf{e}(\alpha) - 1}$. Since $\mathbf{e}(\alpha) - 1 \neq 0$, letting $N \to \infty$ proves the lemma.

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Irrational a

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Problem 8 (Irrational rotation is ergodic).

Show that the rotation of \mathbb{T} by an irrational number α is ergodic with respect to λ .

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Irrational a

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Then we have $\lim_{N} \mathbf{A}_{n \in [N]_0} \mathbf{e}(n\alpha) = 0$.

The proof is an application of the geometric sequence sum-formula, according to which

Problem 8 (Irrational rotation is ergodic).

Hint:

Use first lemma 1 with $\alpha = p\alpha$ for every $p \in \mathbb{Z} \setminus \{0\}$, then use the density of the trigonometric polynomials in $L^1(\mathbb{T},\lambda)$.

 $\mathbf{A}_{n \in [N]_0} \mathbf{e}(n\alpha) = \frac{1}{N} \frac{\mathbf{e}(N\alpha) - 1}{\mathbf{e}(\alpha) - 1}$. Since $\mathbf{e}(\alpha) - 1 \neq 0$, letting $N \to \infty$ proves the lemma.

Show that the rotation of \mathbb{T} by an irrational number α is ergodic with respect to λ .

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$$\mathcal{T}f(x) := f(\mathbf{T}x) \quad \text{for every} \quad f \in \mathbf{L}^2$$
 (16)

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$$\mathcal{P}f \coloneqq \lim_{\mathbf{N}} \mathbf{A}_{n \in [\mathbf{N}]_0} \mathcal{T}^n f \tag{17}$$

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Let (X, \mathbf{m}, T) . The main observation (by *Koopman*) is that the linear transformation \mathcal{T} of L^2 defined by

$$\mathcal{T}f(x) := f(Tx) \quad \text{for every} \quad f \in L^2$$
 (16)

is a linear **isometry**, meaning that $||\mathcal{T}f||_{L^2} = ||f||_{L^2}$. Since L² is a Hilbert space, the following is a

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$$\mathcal{P}f = \lim_{N} \mathbf{A}_{n \in [N]_0} \mathcal{T}^n f \tag{17}$$

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$$\mathcal{T}f(x) := f(Tx) \quad \text{for every} \quad f \in L^2$$
 (16)

is a linear **isometry**, meaning that $\|\mathcal{T}f\|_{L^2} = \|f\|_{L^2}$. Since L^2 is a Hilbert space, the following is a general version of what we need.

Theorem 2 (Mean ergodic theorem for Hilbert space).

Let \mathcal{T} be a **contraction** of a Hilbert space H, so $\|\mathcal{T}f\|_{H} \leq \|f\|_{H}$.

Then

$$\mathcal{P}f := \lim_{\mathbf{N}} \mathbf{A}_{n \in [\mathbf{N}]_0} \mathcal{T}^n f \tag{17}$$

exists in the norm of H for every $f \in H$. The transformation \mathcal{P} is \mathcal{T} invariant: $\mathcal{TP} = \mathcal{P}$.

Recall that $[N]_0 = \{0, 1, ..., N-1\}.$

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$$\lim_{\substack{N \\ n \in [N]_0}} A \mathcal{T}^n f = 0 \quad \text{for} \quad f \in \mathbb{C}$$
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Hilbert space proof

The idea of Riesz for proving the mean ergodic theorem is that two classes of functions in H, for which the theorem is close to obvious, happen to span all of H.

The first class of functions is the γ invariant ones: $\gamma_f = f$. Denoting the set of γ -invariant functions by I, we have

$$\lim_{N} \mathbf{A}_{n \in [N]_{0}} \mathcal{T}^{n} f = f \quad \text{for} \quad f \in I$$
 (18)

The other set of functions is the set C of **coboundaries**; these are functions f which can be written in the form $f = \mathcal{T}g - g$ for some $g \in H$. For such an f we have a collapsing sum:

 $\sum_{n\in[N]_0} \mathcal{T}^n f = \mathcal{T}^N g - g$, thus

$$\lim_{N} \mathbf{A}_{n \in [N]_0} \mathcal{T}^n f = 0 \quad \text{for} \quad f \in \mathbb{C}$$
 (19)

The punchline is the observation that H is the orthogonal sum of I and \overline{C} , H = I \pm \overline{C} . What needs so be proven is that if f is orthogonal to every coboundary then it must be \mathcal{T} -invariant. A picture will make this obviously true: (next slide, please)

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Hilbert space proof

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The first class of functions is the \mathcal{T} invariant ones: $\mathcal{T}f = f$. Denoting the set of \mathcal{T} -invariant functions by I, we have

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The idea of Riesz for proving the mean ergodic theorem is that two classes of functions in H, for which the theorem is close to obvious, happen to span all of H.

The first class of functions is the \mathcal{T} invariant ones: $\mathcal{T}f = f$. Denoting the set of \mathcal{T} -invariant functions by I, we have

$$\lim_{N} \mathbf{A}_{n \in [N]_0} \mathcal{T}^n f = f \quad \text{for} \quad f \in I$$
 (18)

The other set of functions is the set C of **coboundaries**; these are functions f which can be written in the form $f = \mathcal{T}g - g$ for some $g \in H$. For such an f we have a collapsing sum:

$$\sum_{n\in[N]_0} \mathcal{T}^n f = \mathcal{T}^N g - g$$
, thus

$$\lim_{N} \mathbf{A}_{n \in [N]_0} \mathcal{T}^n f = 0 \quad \text{for} \quad f \in C$$
 (19)

The punchline is the observation that H is the orthogonal sum of I and \overline{C} , $H = I \perp \overline{C}$. What needs to be proven is that if f is orthogonal to every coboundary then it must be \mathcal{T} -invariant. A picture will make this obviously true: (next slide, please)

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Hilbert space proof 2

Again, what we are proving is that if f is orthogonal to every coboundary, then it must be \mathcal{T} -invariant, so $\mathcal{T}f = f$.

If $f \perp C$ then, in particular, $f \perp \mathcal{T}f - f$. Here is the picture.



In this right angle triangle, if the length of $\mathcal{T}f=f$ was positive then, by the Pythagorean theorem. $\mathcal{T}f$ would be **strictly longer** than the length of f, but \mathcal{T} is an contraction, so $\|\mathcal{T}f\| \leq \|f\|$. So we must have $\|\mathcal{T}f-f\|=0$, implying $\mathcal{T}f=f$.

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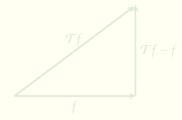
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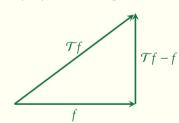
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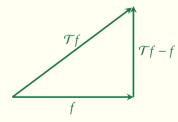
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$$(\mathcal{U}^n f, f) = \int_{\mathbb{T}} \mathbf{e}^{-n} \, \mathrm{d}\mu \quad \text{for every} \quad n \in \mathbb{Z}$$
 (20)

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Lemma 2 (Herglotz's spectral theorem (baby spectral theorem)). Let \mathcal{U} be a unitary transformation of the Hilbert space H, and let $f \in H$.

Then there is a Borel measure $\mu = \mu_f$ on \mathbb{T} so that

$$(\mathcal{U}^n f, f) = \int_{\mathbb{T}} \mathbf{e}^{-n} \, \mathrm{d}\mu \quad \text{for every} \quad n \in \mathbb{Z}$$
 (20)

Note that $\int_{\mathbb{T}} \mathbf{e}^{-n} d\mu = \hat{\mu}(n)$.

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$$(a, m, c, c)$$
 $\int_{-\infty}^{\infty} -n ds$

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Recall:
$$(\mathcal{U}^n f, f) = (f, \mathcal{U}^{-n} f) = \int_{\mathbb{T}} \mathbf{e}^{-n} d\mu_f$$

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Recall:
$$(\mathcal{U}^n f, f) = (f, \mathcal{U}^{-n} f) = \int_{\mathbb{T}} \mathbf{e}^{-n} d\mu_f$$
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Problem 9

Show (or just observe) the following for $f \in H$:

- (a) $||f||_{H}^{2} = \mu_{f} \mathbb{T}$.

Hint:

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- (a) $||f||_{H}^{2} = \mu_{f} \mathbb{T}$.
- (b) $\|\mathbf{A}_{n\in[\mathbb{N}]} \mathcal{U}^n f\|_{\mathbf{H}}^2 = \int_{\mathbb{T}} |\mathbf{A}_{n\in[\mathbb{N}]} \mathbf{e}^{-n}|^2 d\mu$.

Hint:

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- (c) $\lim_{N} \|\mathbf{A}_{n \in [N]} \mathcal{U}^{n} f\|_{\mathbf{H}}^{2} = \mu_{f} \{ 0 \}.$

Hint:

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- (d) $\lim_{N} \mathbf{A}_{n \in [N]} \mathcal{U}^n f$ exists (yes, prove the mean ergodic theorem!)

Hint:

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Hint:

- (a) Observation!

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Hint:

- (a) Observation!
- (b) Calculation using baby spectral theorem

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- (a) Observation!
- (b) Calculation using baby spectral theorem
- (c) Calculation using previous part.

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Hint:

- (a) Observation!
- (b) Calculation using baby spectral theorem

(d) Show that $(\mathbf{A}_{n \in [N]} \mathcal{U}^n f)_{N \in \mathbb{N}}$ is a Cauchy sequence in $\|\cdot\|_{H}$.

- (c) Calculation using previous part.

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$$\lim_{N \to \infty} \mathbf{A} e(t_n \alpha) \text{ exists for every } \alpha \in \mathbb{R}$$
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Definition 2 (Norm-good times).

Let (X, \mathbf{m}, T) be a mps. We say the time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ is **norm-good for** (X, \mathbf{m}, T) if $\lim_{N} \mathbf{A}_{n \in [N]} f \circ T^{t_n}$ exists in $\|\|_{L^2(X)}$ for every $f \in L^2(X)$.

We say the time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ is **norm-good** if it's norm-good for every mps.

$$\lim_{N \to \infty} \mathbf{A}_{n \in [N]} \mathbf{e}(t_n \alpha) \quad \text{exists for every} \quad \alpha \in \mathbb{R}$$
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We say the time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ is **norm-good** if it's norm-good for every mps.

Theorem 3 $((t_n)$ is norm-good iff it's norm good for every rotation).

The time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ is norm-good iff it's norm good for every rotation of the torus \mathbb{T} :

$$\lim_{N} \mathbf{A}_{n \in [N]} \mathbf{e}(t_n \alpha) \quad \text{exists for every} \quad \alpha \in \mathbb{R}$$
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The time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ is norm-good iff it's norm good for every rotation of the torus \mathbb{T} :

$$\lim_{N} \bigwedge_{n \in [N]} \mathbf{e}(t_n \alpha) \quad \text{exists for every} \quad \alpha \in \mathbb{R}$$
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Hint (to prove theorem 3):

Use baby spectral theorem as you did in problem 9, (d).

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Squares are norm good

Corollary 1 (to theorem 3).

The squares and cubes are norm-good.

I just deal with the squares, leaving the cubes for you to explore.

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Squares are norm good

Corollary 1 (to theorem 3).

The squares and cubes are norm-good.

I just deal with the squares, leaving the cubes for you to explore.

- We need to show that $\lim_{N} \mathbf{A}_{n \in [N]} \mathbf{e}(n^2 \alpha)$ exists for every rotation α .

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Theorem 4 (Pointwise ergodic theorem).

Let (X, \mathbf{m}, T) be a measure preserving system and $f \in L^1(X)$

Then $\lim_{N \to \infty} \mathbf{A}_{n \in [N]_0} f(T^n x)$ exists for **m**-a.e. $x \in X$.

Corollary 2 (Irrational rotations)

Let α be irrational

Then $\lim_{N} \mathbf{A}_{n \in [N]_0} f(\theta + n\alpha) = \int_{\mathbb{T}} f \, d\lambda \text{ for } f \in L^1(\mathbb{T}, \lambda) \text{ and } \lambda\text{-a.e. } \theta$

In case of pointwise ergodic theorems, we are preoccupied with showing the existence of limit. We leave it to norm-convergence to identify the limit.

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Find an L¹-dense class of functions for which convergence holds

Prove a maximal inequality.

The dense class is the same as in case of the mean ergodic theorem: those $f \in L^2(X)$ which can be written as a sum of a T-invariant function and a coboundary. The only question is a coboundary, so when $f = g \circ T - g$ for some $g \in L^2$. Since $\mathbf{A}_{n \in [N]_0} f(T^n x) = \frac{1}{N} \cdot (g(T^N x) - g(x))$.

$$\int_{X} \sum_{N} \left| A_{n \in [N]_0} f(T^n x) \right|^2 \le \sum_{N} \frac{1}{N^2} \cdot 4 \cdot \|g\|_{L^2(X)}^2 < \infty$$
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so $\lim_{N \to [N]_0} f(T^n x) = 0$ for **m**-a.e. $x \in X$.

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so $\lim_{N} \mathbf{A}_{n \in [N]_0} f(T^n x) = 0$ for **m**-a.e. $x \in X$.

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Maximal inequality

$$\mathbf{m}\left\{x \in X : \sup_{\mathbf{N}} \left| \underbrace{\mathbf{A}}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
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Maximal inequality

The pointwise convergence from a dense set of L^1 is extended to all of L^1 by a **maximal** inequality:

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
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 (23)

The function $\sup_{\mathbf{N}} |\mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x)|$ is called the **maximal function of f**. Note that for any $g \in L^1(X)$ we have Markov's inequality $\mathbf{m} \{ x \in X : \sup_{\mathbf{N}} |g(x)| > \eta \} \le \frac{1}{\eta} ||g||_{L^1(X)}$. In general, the maximal function is not integrable, though it satisfies the **weak inequality** in eq. (23).

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Scheme to prove the maximal inequality

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \underbrace{\mathbf{A}}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
 (24)

$$\pi \left\{ j \in \mathbb{Z}_+ : \max_{\mathbf{N} \in [M]_n \in [N]_0} \mathbb{A}_{\mathbf{F}(j+n)} > 1 \right\} \le \sum_{j} \mathbb{F}(j)$$
 (25)

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Scheme to prove the maximal inequality

When proving

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
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we can assume that $f \ge 0$ and $\eta = 1$. The proof of eq. (24) is in two steps:

$$\#\left\{j \in \mathbb{Z}_{+} : \max_{N \in [M]} \underbrace{\mathbf{A}}_{n \in [N]_{0}} F(j+n) > 1\right\} \leq \sum_{j} F(j)$$

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we can assume that $f \ge 0$ and $\eta = 1$. The proof of eq. (24) is in two steps:

▶ Prove the special case when the mps is $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ with the counting measure and T is the shift by 1, so Tj = j + 1. We thus first prove

$$\#\left\{j \in \mathbb{Z}_{+} : \max_{\mathbf{N} \in [\mathbf{M}]} \mathbf{A}_{n \in [\mathbf{N}]_{0}} \mathbf{F}(j+n) > 1\right\} \leq \sum_{j} \mathbf{F}(j)$$

$$(25)$$

for $F : \mathbb{Z}_+ \to \mathbb{R}_+$ of finite support and $M \in \mathbb{N}$.

Use Calderón's transference to show the inequality in any mps.

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Scheme to prove the maximal inequality

When proving

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
 (24)

we can assume that $f \ge 0$ and $\eta = 1$. The proof of eq. (24) is in two steps:

▶ Prove the special case when the mps is $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ with the counting measure and T is the shift by 1, so Tj = j + 1. We thus first prove

$$\#\left\{j \in \mathbb{Z}_{+} : \max_{\mathbf{N} \in [\mathbf{M}]} \mathbf{A}_{n \in [\mathbf{N}]_{0}} \mathbf{F}(j+n) > 1\right\} \le \sum_{j} \mathbf{F}(j)$$

$$\tag{25}$$

for $F : \mathbb{Z}_+ \to \mathbb{R}_+$ of finite support and $M \in \mathbb{N}$.

▶ Use Calderón's **transference** to show the inequality in any mps.

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$$\mathbf{n}\{x \in X : T^q x = x \text{ for some } q \in \mathbb{N}\} = 0$$
 (26)

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$$\mathbf{m}\{x \in X : T^q x = x \text{ for some } q \in \mathbb{N}\} = 0$$

$$\in \mathbb{N} \} = 0 \tag{26}$$

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A transformation T of a measure space (X, m) is aperiodic if

$$\mathbf{m}\{ x \in X : T^q x = x \text{ for some } q \in \mathbb{N} \} = 0$$

Recall that earlier you constructed a time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ so that for every $q \in \mathbb{N}$ $\limsup_{N} \mathbf{A}_{n \in [N]} \mathbb{1}_{\{0\}}(r + t_n) = 1 \text{ for every } r \in [q].$

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Recall that earlier you constructed a time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ so that for every $q \in \mathbb{N}$ $\limsup_{\mathbf{N}} \mathbf{A}_{n \in [\mathbf{N}]} \mathbb{1}_{\{0\}}(r + t_n) = 1$ for every $r \in [q]$.

Problem 10 (Bad time (Krengel)).

Let (X, \mathbf{m}, T) be an aperiodic mps and $\varepsilon > 0$.

Show that there is $A \subset X$ with $\mathbf{m} A < \varepsilon$ and $\limsup_{N} \mathbf{A}_{n \in [N]} \mathbb{1}_{A}(T^{t_n}x) = 1$ for a.e. $x \in X$.

Hint:

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See the appendix.

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A transformation T of a measure space (X, m) is aperiodic if

$$\mathbf{m} \{ x \in X : T^q x = x \text{ for some } q \in \mathbb{N} \} = 0$$

Recall that earlier you constructed a time $\mathbf{t} = (t_n)_{n \in \mathbb{N}}$ so that for every $q \in \mathbb{N}$ $\limsup_{N} \mathbf{A}_{n \in [N]} \mathbb{1}_{\{0\}}(r + t_n) = 1 \text{ for every } r \in [q].$

Problem 10 (Bad time (Krengel)).

Let (X, \mathbf{m}, T) be an aperiodic mps and $\varepsilon > 0$. Show that there is $A \subset X$ with $\mathbf{m} A < \varepsilon$ and $\limsup_{N \to \infty} \mathbf{A}_{n \in [N]} \mathbb{1}_A(\mathbf{T}^{t_n} x) = 1$ for a.e. $x \in X$.

Hint:

See the appendix.

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Maximal inequality on \mathbb{Z}_+

$$\#\left\{j\in\mathbb{Z}_{+}:\max_{\mathrm{N}\in[\mathrm{M}]} \mathbf{A}_{n\in[\mathrm{N}]_{0}} \mathrm{F}(j+n)>1\right\} \leq \sum_{j} \mathrm{F}(j) \quad \text{for } \mathrm{F}:\mathbb{Z}_{+} \to \mathbb{R}_{+} \text{ of finite support and } \mathrm{M}\in\mathbb{N}$$

$$_{1}.F(i+n) > N$$

an
$$N_1 \in [M]$$
 we have $\Sigma_{n \in I}$

$$N_1 < \sum_{i \in [i, i, i+N_1-1]} J_i$$

$$j_1 + N_1 - 1$$

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Maximal inequality on \mathbb{Z}_+

We want to prove

$$\#\left\{j \in \mathbb{Z}_{+} : \max_{N \in [M]} \mathbf{A}_{n \in [N]_{0}} F(j+n) > 1\right\} \leq \sum_{j} F(j) \quad \text{for } F : \mathbb{Z}_{+} \to \mathbb{R}_{+} \text{ of finite support and } M \in \mathbb{N}$$

$$N \in [M]$$
 $n \in [N]_0$

▶ Let
$$S = S_1 := \{j \in \mathbb{Z} \}$$

$$j \in [j_1, j_1]$$

$$[j_1,j_1+N]$$

$$\mathbb{P}(j)$$

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Maximal inequality on \mathbb{Z}_+

We want to prove

$$\#\left\{j \in \mathbb{Z}_{+} : \max_{N \in [M]} \mathbf{A}_{n \in [N]_{0}} F(j+n) > 1\right\} \leq \sum_{j} F(j) \quad \text{for } F : \mathbb{Z}_{+} \to \mathbb{R}_{+} \text{ of finite support and } M \in \mathbb{N}$$
(27)

Let $S = S_1 := \{ j \in \mathbb{Z}_+ : \sum_{n \in [N]_0} F(j+n) > N \text{ for some } N \in [M] \}$. Let j_1 be the smallest element of S₁. Then for an N₁ \in [M] we have $\sum_{n \in [N_1]_0} F(j_1 + n) > N_1$ which is

> $N_1 < \sum_{i=1}^{n} F(j)$ $i \in [i_1, i_1 + N_1 - 1]$

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▶ Let j_2 be the smallest element of $S_2 := S_1 \setminus [j_1, j_1 + N_1 - 1]$. We then have for some $N_2 \in [M]$,

$$N_2 < \sum_{i=1,\dots,N-1}$$

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Maximal inequality on \mathbb{Z}_+

$$\#\left\{j\in\mathbb{Z}_{+}:\max_{\mathrm{N}\in[\mathrm{M}]}\mathbf{A}_{n\in[\mathrm{N}]_{+}}\mathrm{F}(j+n)>1\right\}\leq\sum_{i}\mathrm{F}(j)\quad\text{for }\mathrm{F}:\mathbb{Z}_{+}\to\mathbb{R}_{+}\text{ of finite support and }\mathrm{M}\in\mathbb{N}$$

$$\int -\frac{2}{j}$$

 $i \in [i_1, i_1 + N_1 - 1]$

► Let $S = S_1 := \{ j \in \mathbb{Z}_+ : \sum_{n \in [N]_0} F(j+n) > N \text{ for some } N \in [M] \}$. Let j_1 be the

smallest element of S₁. Then for an N₁ \in [M] we have $\sum_{n \in [N_1]_0} F(j_1 + n) > N_1$ which is

 $N_1 < \sum_{i=1}^{n} F(j)$

(28)

▶ Let j_2 be the smallest element of $S_2 := S_1 \setminus [j_1, j_1 + N_1 - 1]$. We then have for some $N_2 \in [M]$,

 $N_2 < \sum_{j \in [j_2, j_2 + N_2 - 1]} F(j)$

See the picture on the next slide.

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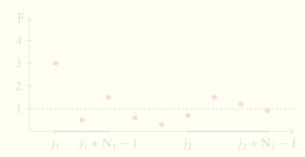
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Maximal inequality on \mathbb{Z}_+



$$fS \leq N_1 + N_2 + \dots + N_k$$

$$< \sum_{l \in [k]} \sum_{j \in [j_l, j_l + N_l - 1]} F(j)$$

$$\leq \sum_{j} F(j)$$

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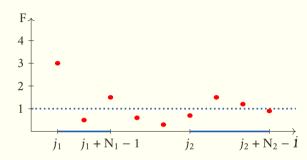
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Maximal inequality on \mathbb{Z}_+



$$S \leq N_1 + N_2 + \dots + N_k$$

$$< \sum_{l \in [k]} \sum_{j \in [j_l, j_l + N_l - 1]} F(j)$$

$$\leq \sum_{j} F(j)$$

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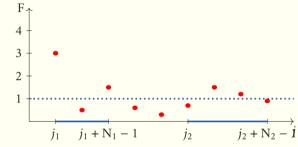
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Maximal inequality on \mathbb{Z}_+



We have $N_1 < \sum_{i \in [i,j,j+N,-1]} F(j)$ and $N_2 < \sum_{i \in [i,j,j+N,-1]} F(j)$. Continuing this way, we will stop after k steps when $S \subset \bigcup_{l \in [k]} [j_l + 1, j_l + N_l]$, where the intervals are disjoint, thus

after
$$k$$
 steps when $S \subset \bigcup_{l \in [k]} [j_l+1, j_l+N_l]$, where the inter
$$*S \leq N_1+N_2+\cdots+N_k$$

$$< \sum_{l \in [k]} \sum_{j \in [j_l, j_l+N_l-1]} F(j_l)$$

$$\leq \sum_j F(j_l)$$

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$$F(i) = f(T^i x)$$
 for $i \in$

$$\sum_{j \in [J-M]_0} \mathbf{1}_{\mathbf{S}}(\mathbf{T}^j x) \le \mathbb{E} \left\{ j \in [J]_0 : \max_{\mathbf{N} \in [M]} \mathbf{A}_{n \in [N]_0} \mathbf{F}(j+n) > 1 \right\}$$

$$\le \sum_{j \in [J]_0} \mathbf{F}(j)$$

$$= \sum_{j \in [J]_0} f(\mathbf{T}^j x)$$

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The transference

Denote $S := \{ x \in X : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} f(T^n x) > 1 \}$. We want to show $\mathbf{m} S \leq \int_Y f d\mathbf{m}$.

$$F(i) = f(T^{i}x)$$
 for $i \in$

$$\sum_{j \in [J-M]_0} \mathbb{1}_{S}(T^j x) \le \# \left\{ j \in [J]_0 : \max_{N \in [M]} \bigwedge_{n \in [N]_0} F(j+n) > 1 \right\}$$

$$\le \sum_{j \in [J]_0} F(j)$$

$$= \sum_{j \in [J]_0} f(T^j x)$$

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The transference

Denote $S := \{ x \in X : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} f(T^n x) > 1 \}$. We want to show $\mathbf{m} S \leq \int_Y f d\mathbf{m}$. Fix $x \in X$ and let $J \in \mathbb{N}$, J > M and define $F : [J]_0 \to \mathbb{R}_+$ by

$$F(i) = f(T^{i}x)$$
 for $i \in [1]_0$

$$\sum_{j \in [J-M]_0} \mathbb{1}_{S}(T^j x) \le \# \begin{cases} j \in [J]_0 : \max_{N \in [M]} \bigwedge_{n \in [N]_0} F(j+n) \\ \le \sum_{j \in [J]_0} F(j) \\ = \sum_{j \in [J]_0} f(T^j x) \end{cases}$$

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$$F(i) = f(T^{i}x)$$
 for $i \in [1]_0$

Observe that if $j \in [J - M]_0$, then $\mathbf{A}_{n \in [N]_0} f(T^{j+n} x) = \mathbf{A}_{n \in [N]_0} F(j+n)$, hence if $T^j x \in S$ for

 $j \in [J-M]_0$, then $j \in \{j \in [J]_0 : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} F(j+n) > 1\}$. Thus

$$\sum_{j \in [J-M]_0} \mathbb{1}_{S}(T^j x) \le \# \left\{ j \in [J]_0 : \max_{N \in [M]} \underset{n \in [N]_0}{\mathbf{A}} F(j+n) > 1 \right\}$$
$$\le \sum_{j \in [J]_0} F(j)$$
$$= \sum_{j \in [J]_0} f(T^j x)$$

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The transference

Denote $S := \{ x \in X : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} f(T^n x) > 1 \}$. We want to show $\mathbf{m} S \leq \int_Y f d\mathbf{m}$. Fix $x \in X$ and let $J \in \mathbb{N}$, J > M and define $F : [J]_0 \to \mathbb{R}_+$ by

$$F(j) = f(T^j x)$$
 for $j \in [J]_0$

Observe that if $j \in [J - M]_0$, then $\mathbf{A}_{n \in [N]_0} f(T^{j+n} x) = \mathbf{A}_{n \in [N]_0} F(j+n)$, hence if $T^j x \in S$ for $j \in [J-M]_0$, then $j \in \{j \in [J]_0 : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} F(j+n) > 1\}$. Thus

$$\sum_{j \in [J-M]_0} \mathbb{1}_{S}(T^{j}x) \le \# \left\{ j \in [J]_0 : \max_{N \in [M]} A_{n \in [N]_0} F(j+n) > 1 \right\}$$

$$\le \sum_{j \in [J]_0} F(j)$$

$$= \sum_{j \in [J]_0} f(T^{j}x)$$

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The transference

Denote $S := \{ x \in X : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} f(T^n x) > 1 \}$. We want to show $\mathbf{m} S \leq \int_Y f d\mathbf{m}$. Fix $x \in X$ and let $J \in \mathbb{N}$, J > M and define $F : [J]_0 \to \mathbb{R}_+$ by

Observe that if $j \in [J - M]_0$, then $\mathbf{A}_{n \in [N]_0} f(T^{j+n} x) = \mathbf{A}_{n \in [N]_0} F(j+n)$, hence if $T^j x \in S$ for

$$F(j) = f(T^j x)$$
 for $j \in [J]_0$

 $\sum_{j \in [1-M]_0} \mathbb{1}_{S}(T^j x) \le \# \left\{ j \in [J]_0 : \max_{N \in [M]} A_{n \in [N]_0} F(j+n) > 1 \right\}$ $\leq \sum_{i} F(j)$

 $i \in [I]_0$

 $j \in [I]_0$

 $=\sum_{i} f(T^{j}x)$

 $j \in [J-M]_0$, then $j \in \{j \in [J]_0 : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} F(j+n) > 1\}$. Thus

Integrate both ends in x and use that T preserves measure to get (J - M) m $S \le J \int_V f dm$. Divide

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Denote $S := \{ x \in X : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} f(T^n x) > 1 \}$. We want to show $\mathbf{m} S \leq \int_Y f d\mathbf{m}$. Fix $x \in X$ and let $J \in \mathbb{N}$, J > M and define $F : [J]_0 \to \mathbb{R}_+$ by

by J and let it go to ∞ to get $\mathbf{m} S \leq \int_{V} f d\mathbf{m}$.

 $F(i) = f(T^{i}x)$ for $i \in [1]_0$

 $\sum_{j \in [1-M]_0} \mathbb{1}_{S}(T^j x) \le \# \left\{ j \in [J]_0 : \max_{N \in [M]} A_{n \in [N]_0} F(j+n) > 1 \right\}$

Integrate both ends in x and use that T preserves measure to get (J - M) **m** $S \le J \int_V f d\mathbf{m}$. Divide

Observe that if $j \in [J - M]_0$, then $\mathbf{A}_{n \in [N]_0} f(T^{j+n} x) = \mathbf{A}_{n \in [N]_0} F(j+n)$, hence if $T^j x \in S$ for

 $j \in [J-M]_0$, then $j \in \{j \in [J]_0 : \max_{N \in [M]} \mathbf{A}_{n \in [N]_0} F(j+n) > 1\}$. Thus

 $\leq \sum_{i} F(j)$

 $=\sum_{i}f(T^{j}x)$

 $j \in [1]_0$

 $j \in [J]_0$

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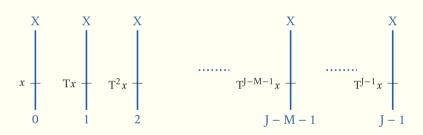


Figure: Transference from $[J]_0$ to any measure preserving system

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$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \underbrace{\mathbf{A}}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
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Maximal inequality and a.e. convergence

How does the maximal inequality

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
 (31)

and a.e. convergence for an L¹-dense class imply a.e. convergence for all of L¹?

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Maximal inequality and a.e. convergence

How does the maximal inequality

$$\mathbf{m}\left\{x \in X : \sup_{\mathbf{N}} \left| \mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
 (31)

and a.e. convergence for an L¹-dense class imply a.e. convergence for all of L¹?

- Fix $f \in L^1(X)$, let $\varepsilon > 0$, and let g be from dense class for which $\lim_N \mathbf{A}_{n \in [N]_0} g(T^n x)$ exists a.e. and approximates f ε -closely: f = g + h where $||h||_{L^1(X)} < \varepsilon$.
- Denoting by $\omega F(x)$ the **oscillation** of the sequence $(\mathbf{A}_{n \in [N]_0} F(T^n x))_{n \in \mathbb{N}}$, so $\omega F(x) = \lim \sup_{K,N} |\mathbf{A}_{n \in [N]_0} F(T^n x) \mathbf{A}_{n \in [K]_0} F(T^n x)|$, we have $\omega f(x) = \omega h(x)$.
- Since $\omega h(x) \le 2 \sup_{\mathbf{N}} |\mathbf{A}_{n \in [\mathbf{N}]_0} h(x)|$, by the maximal inequality, we get $\mathbf{m} \{ x \in \mathbf{X} : \omega f(x) > \eta \} \le \mathbf{m} \{ x \in \mathbf{X} : 2 \sup_{\mathbf{N}} |\mathbf{A}_{n \in [\mathbf{N}]_0} h|(x) > \eta \} \le \frac{2}{\pi} ||h||_{\mathbf{L}^1(\mathbf{X})}.$
- ▶ Since $||h||_{L^1(X)} < \varepsilon$ and $\varepsilon > 0$ was arbitrary, we get that $\mathbf{m}\{x \in X : \mathbf{\omega} f(x) > \eta\} = 0$. Since this is true for every $\eta > 0$, we get that $\mathbf{\omega} f(x) = 0$ for \mathbf{m} -a.e. $x \in X$.

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Maximal inequality and a.e. convergence

How does the maximal inequality

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{A}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
 (31)

and a.e. convergence for an L¹-dense class imply a.e. convergence for all of L¹?

- ► Fix $f \in L^1(X)$, let $\varepsilon > 0$, and let g be from dense class for which $\lim_N \mathbf{A}_{n \in [N]_0} g(T^n x)$ exists a.e. and approximates f ε -closely: f = g + h where $||h||_{L^1(X)} < \varepsilon$.
- ▶ Denoting by ω F(x) the **oscillation** of the sequence $(\mathbf{A}_{n \in [N]_0} \mathrm{F}(\mathrm{T}^n x))_{n \in \mathbb{N}}$, so ω F(x) = $\limsup_{K,N} |\mathbf{A}_{n \in [N]_0} \mathrm{F}(\mathrm{T}^n x) \mathbf{A}_{n \in [K]_0} \mathrm{F}(\mathrm{T}^n x)|$, we have ω f(x) = ω h(x).
- Since $\omega h(x) \le 2 \sup_{\mathbb{N}} |\mathbf{A}_{n \in [\mathbb{N}]_0} h(x)|$, by the maximal inequality, we get $\mathbf{m} \{ x \in \mathbb{X} : \omega f(x) > \eta \} \le \mathbf{m} \{ x \in \mathbb{X} : 2 \sup_{\mathbb{N}} |\mathbf{A}_{n \in [\mathbb{N}]_0} h|(x) > \eta \} \le \frac{2}{n} ||h||_{L^1(\mathbb{X})}.$
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Maximal inequality and a.e. convergence

How does the maximal inequality

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{\Lambda}_{n \in [\mathbf{N}]_0} f(\mathbf{T}^n x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^1(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^1(\mathbf{X}), \eta > 0$$
 (31)

and a.e. convergence for an L^1 -dense class imply a.e. convergence for all of L^1 ?

- Fix $f \in L^1(X)$, let $\varepsilon > 0$, and let g be from dense class for which $\lim_N \mathbf{A}_{n \in [N]_0} g(T^n x)$ exists a.e. and approximates f ε -closely: f = g + h where $||h||_{L^1(X)} < \varepsilon$.
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How does the maximal inequality

$$\mathbf{m}\left\{x \in \mathbf{X} : \sup_{\mathbf{N}} \left| \mathbf{A}_{\mathbf{x} \in [\mathbf{N}]_{+}} f(\mathbf{T}^{n} x) \right| > \eta \right\} \le \frac{1}{\eta} \|f\|_{\mathbf{L}^{1}(\mathbf{X})} \quad \text{for every} \quad f \in \mathbf{L}^{1}(\mathbf{X}), \eta > 0$$

and a.e. convergence for an
$$L^1$$
-dense class imply a.e. convergence for all of L^1 ?

► Fix $f \in L^1(X)$, let $\varepsilon > 0$, and let g be from dense class for which $\lim_N \mathbf{A}_{n \in [N]_0} g(T^n x)$ exists a.e. and approximates f ε -closely: f = g + h where $||h||_{L^1(X)} < \varepsilon$.

- ▶ Denoting by ω F(x) the **oscillation** of the sequence $(A_{n ∈ [N]_0} F(T^n x))_{n ∈ \mathbb{N}}$, so ω F(x) = $\limsup_{K \to \mathbb{N}} |A_{n ∈ [N]_0} F(T^n x) A_{n ∈ [K]_0} F(T^n x)|$, we have ω f(x) = ω h(x).
- Since $\boldsymbol{\omega} h(x) \leq 2 \sup_{\mathbf{N}} |\mathbf{A}_{n \in [\mathbf{N}]_0} h(x)|$, by the maximal inequality, we get $\mathbf{m} \{ x \in \mathbf{X} : \boldsymbol{\omega} f(x) > \eta \} \leq \mathbf{m} \{ x \in \mathbf{X} : 2 \sup_{\mathbf{N}} |\mathbf{A}_{n \in [\mathbf{N}]_0} h|(x) > \eta \} \leq \frac{2}{n} ||h||_{L^1(\mathbf{X})}.$
- ► Since $||h||_{L^1(X)} < \varepsilon$ and $\varepsilon > 0$ was arbitrary, we get that $\mathbf{m}\{x \in X : \mathbf{\omega} f(x) > \eta\} = 0$. Since this is true for every $\eta > 0$, we get that $\mathbf{\omega} f(x) = 0$ for \mathbf{m} -a.e. $x \in X$.

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Rohlin's interval

Theorem 5 (Rohlin's interval with error).

Let T be an invertible, bimeasurable, measure preserving, aperiodic transformation of the measure space (X, \mathbf{m}) . Let $[a, b] \subset \mathbb{Z}$ be an interval and let $\varepsilon > 0$.

Then there is a measurable B \subset X so that the sets in the family $\{T^jB: j \in [a,b]\}$ are pairwise disjoint and $\mathbf{m} \left(\bigcup_{j \in [a,b]} T^jB\right)^c < \varepsilon$.

The set $E := (\bigcup_{j \in [a,b]} T^j B)^c$ is called the **error set**, and we talk about *Rohlin's interval* [a,b] with error ε . A bit more casually, we say we represent the interval [a,b] in a mps. In the literature they talk about *Rohlin's tower of height* H with error ε and they mean representing the interval [0,H).

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Theorem 5 (Rohlin's interval with error).

Let T be an invertible, bimeasurable, measure preserving, aperiodic transformation of the measure space (X, \mathbf{m}) . Let $[a, b] \subset \mathbb{Z}$ be an interval and let $\varepsilon > 0$.

Then there is a measurable B \subset X so that the sets in the family $\{T^jB: j \in [a,b]\}$ are pairwise disjoint and $\mathbf{m} \left(\bigcup_{i \in [a,b]} T^i B \right)^c < \varepsilon$.

The set $E := (\bigcup_{i \in [a,b]} T^i B)^c$ is called the **error set**, and we talk about *Rohlin's interval* [a,b] with *error* ε . A bit more casually, we say we represent the interval [a, b] in a mps. In the literature they talk Basecamp for convergence Mátá Wiardl

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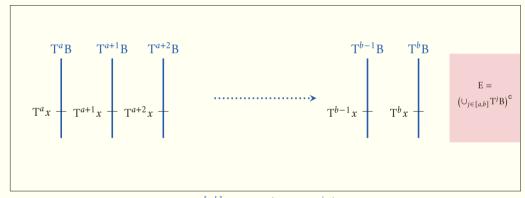
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Rohlin's interval [a, b] and error:



$$X = T^{[a,b]}B \cup E = (\bigcup_{i \in [a,b]} T^{i}B) \cup E$$

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Show that there is $A \subset X$ with $\mathbf{m} A < \varepsilon$ and $\limsup_{N} \mathbf{A}_{n \in [N]} \mathbb{1}_A(T^{t_n}x) = 1$ for a.e. $x \in X$.

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- ▶ We let F be the indicator of the set $\{sq : s \in [0, p)\} \subset \mathbb{Z}$. Denoting J = pq, for every $j \in [0, J M)$ we have $\max_{\{N: t_N \in [M]\}} \mathbf{A}_{n \in [N]} F(j + t_n) > 1 \eta$.
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